Removal of 2nd term

1) Consider the Polynomial equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0 \quad \text{---(i)}$$

To remove the 2nd term to ~~term~~ diminish the roots of the equation by h where h is given by $h = -\dfrac{a_1}{a_0 n}$

2) Remove the 2nd term of the equation $x^3+6x^2+12x-9=0$

$\Rightarrow$ given equation,

$$x^3+6x^2+12x-9=0 \quad \text{---(i)}$$

where $a_0 = 1$, $n = 3$ and $a_1 = 6$

$$\therefore h = -\frac{a_1}{a_0 n} = -\frac{6}{3} = -2$$

Let, $y = x - h = x-(-2) = x+2$

$\therefore x = y-2$

$\therefore$ from (i),

$$(y-2)^3 + 6(y-2)^2 + 12(y-2) - 9 = 0$$

$$y^3 + y^2(-6+6) + y(+12-24+12) + (-8+24-24-9) = 0$$

or, $y^3 + y^2(-6+6) + y(+12-24+12)$

or, $y^3 - 17 = 0$ ---(ii)

(Synthetic division):—

$$\begin{array}{r|rrrr}
-2 & 1 & 6 & 12 & -9 \\
 & & -2 & -8 & -8 \\
\hline
 & 1 & 4 & 4 & -17 \\
 & & -2 & -4 & \\
\hline
 & 1 & 2 & 0 & \\
 & & -2 & & \\
\hline
 & 1 & 0 & & \\
\hline
 & 1 & & &
\end{array}$$

$y^3 + 0 \cdot y^2 + 0 \cdot y - 17 = 0$

or, $y^3 - 17 = 0$

3) Remove the second term of the equation $x^4 + 8x^3 + x - 5 = 0$

$\Rightarrow$ given equation,

$$x^4 + 8x^3 + x - 5 = 0 \quad \text{---(i)}$$

Where, $a_0 = 1$, $a_1 = 8$, $n = 4$

$$\therefore h = -\frac{a_1}{a_0 n} = -\frac{8}{4} = -2$$

Let, $y = x-(-2) = x+2$

or, $x = y-2$

~~from (i), $(y-2)^4$~~

$$\begin{array}{r|rrrrr}
-2 & 1 & 8 & 0 & 1 & -5 \\
 & & -2 & -12 & 24 & -50 \\
\hline
 & 1 & 6 & -12 & 25 & -55 \\
 & & -2 & -8 & 40 & \\
\hline
 & 1 & 4 & -20 & 65 & \\
 & & -2 & -4 & & \\
\hline
 & 1 & 2 & -24 & & \\
 & & -2 & & & \\
\hline
 & 1 & 0 & & & \\
\hline
 & 1 & & & &
\end{array}$$

$\therefore y^4 - 24y^2 + 65y - 55 = 0$

4) **Solution of cubic equation : Cardan's Method**

The standard form of cubic equation,

$$z^3 + 3Hz + G = 0$$

If $G^2 + 4H^3 > 0$ then Cardan's Method can be applied.

5) Solve the following equation $x^3 - 18x - 35 = 0$.

⇒ $x^3 - 18x - 35 = 0$ — (i)

Here, $H = -6$ and $G = -35$

Now, $G^2 + 4H^3 = (35)^2 + 4(-6)^3 = 361 > 0$

Let, $x = u + v$

$\therefore x^3 = (u+v)^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$

or, $x^3 - 3uvx - (u^3 + v^3) = 0$ — (ii)

Comparing (i) and (ii),

$-3uv = -18$

or, $uv = 6$ — (iii)

$-(u^3 + v^3) = -35$

or, $u^3 + v^3 = 35$ — (iv)

Now, $(u^3 - v^3)^2 = (u^3 + v^3)^2 - 4u^3v^3$

$= (35)^2 - 4(6)^3$ [by (iii) and (iv)]

$= 361$

$\therefore u^3 - v^3 = \pm 19$ — (v)

taking Positive sign, $u^3 = 27$

or, $u = 3, 3\omega, 3\omega^2$

from (iii), $v = \frac{6}{u} = \frac{6}{3}, \frac{6}{3\omega}, \frac{6}{3\omega^2}$

$= 2, \frac{2}{\omega}, \frac{2}{\omega^2}$

$= 2, 2\omega^2, 2\omega$

$\therefore x = u + v = 3 + 2, 3\omega + 2\omega^2, 3\omega^2 + 2\omega = 5 + 2 + 3\omega + 2\omega^2 + 2\omega - 2, 3\omega^2 + 3\omega + 3$

$= 5, \omega - 2, \cancel{\omega^2 - 3}$

4) Solve by Cardon's Method $x^3 - 30x + 133 = 0$

$\Rightarrow$ $x^3 - 30x + 133 = 0$ ——(i)

Here, ~~3H~~

Comparing the equation (i) with standard cubic equation,

$$3H = -30 \quad \text{and} \quad G = 133$$

or, $H = -10$

$\therefore G^2 + 4H^3 = (133)^2 + 4(-10)^3 = 13689 > 0$

$\therefore$ Cardon's method can be applied.

Let, $x = u + v$ $\qquad u^3 + v^3 = -G = -133$

$uv = -H = 10$ ——(ii)

$\cancel{u^3 + v^3 = -G = -133}$

and $u^3 = \dfrac{-G + \sqrt{G^2 + 4H^3}}{2} = \dfrac{-133 + 117}{2} = -8$

$\therefore u = -2, -2\omega, -2\omega^2$

$\therefore$ from (ii), $v = \dfrac{10}{u} = \dfrac{10}{-2}, \dfrac{10}{-2\omega}, \dfrac{10}{-2\omega^2}$

$= -5, -\dfrac{5}{\omega}, -\dfrac{5}{\omega^2} = -5, -5\omega^2, -5\omega$

$\therefore x = u + v = -2-5, -2\omega - 5\omega^2, -2\omega^2 - 5\omega$ $\left[\omega = \dfrac{-1-i\sqrt{3}}{2}\right]$

$= -7, -2\omega - 5\omega^2, -2\omega^2 - 5\omega$

---

1) Let, $f(x) = x^3 + 3px + q$ ——(1)

$\because (x-a)^2$ is a factor of $f(x) = 0$.

$\therefore$ a is the multiple root of $f(x) = 0$ with multiplicity 2.

We have, $f'(x) = 3x^2 + 3p$

$\therefore$ ~~If~~ a is the multiple root with multiplicity 2.

$\therefore f(a) = a^3 + 3pa + q = 0$ ——(i)

and $f'(a) = 3a^2 + 3p = 0$ ——(ii)

or, $p = -a^2$.

$\therefore$ from (i), $a^3 - 3a^3 + q = 0$

or, $q = 2a^3$ $\qquad a^2(2a^2)^2 = a^2(a^2+3p)^2$

or, $q^2 = 4a^6 = $

$3 \dfrac{-3}{-\omega}$

20) Let, $f(x) = x^n - px + r$

$\therefore$ We have, $f'(x) = nx^{n-1} - p$

Let, α is the root of the equation $f(x)=0$ with multiplicity 2.

$\therefore f(\alpha) = \alpha^n - p\alpha + r = 0$ — (i)

and $f'(\alpha) = n\alpha^{n-1} - p = 0$

or, $p = n\alpha^{n-1}$ — (ii)

$\therefore$ from (i), $\alpha^n - n\alpha^n + r = 0$

or, $r = \alpha^n(n-1)$ — (iii)

Now, $4p^n (n-2)^{n-2} = 4(n\alpha^{n-1})^n (n-2)^{n-2}$

$= 4n^n \alpha^{n^2-n} (n-2)^{n-2} = 4n^n \alpha^{n^2-2n} (n-2)^{n-2} \alpha^n \cdot \alpha^n$

$= n^n$

$n^n r^{n-2} = n^n \{\alpha^n (n-1)\}^{n-2} = n^n \alpha^{n^2-2n} (n-1)^{n-2}$

21) Let, $f(x) = x^n - qx^{n-m} + r$

$\therefore f'(x) = nx^{n-1} - (n-m)qx^{n-m-1}$

Let, α is a multiple root of the equation with multiplicity 2

$\therefore f(\alpha) = \alpha^n - q\alpha^{n-m} + r = 0$ — (i)

and $f'(\alpha) = n\alpha^{n-1} - (n-m)q\alpha^{n-m-1} = 0$ — (ii)

or, $q = \dfrac{n\alpha^{n-1}}{(n-m)\alpha^{n-m-1}}$

from (i), $r = q\alpha^{n-m} - \alpha^n = \dfrac{n\alpha^{n-1} \cdot \alpha^{n-m}}{(n-m)\alpha^{n-m-1}} - \alpha^n$

$= \dfrac{n\alpha^{2n-m-1}}{(n-m)\alpha^{n-m-1}} - \alpha^n$

$\therefore$ L.H.S. $= \left\{\dfrac{q}{n}(n-m)\right\}^n = \left(\dfrac{\alpha^{n-1}}{(n-m)\alpha^{n-m-1}}\right)^n (n-m)^n$

$$= \frac{\alpha^{n^2-2n}}{(n-m)^{n^2-nm-n}} \times (n-m)^{n} = \frac{\alpha^{n^2-2n}}{(n-m)^{n^2-nm-2n}}$$

$$R.H.S. = \left\{\frac{n}{m}(n-m)\right\}^{m} = \left(\frac{n\alpha^{2n-m-1}}{(n-m)^{n-m-1}} - \alpha^{n}\right)^{m} ($$

22) $f(x) = x^4 + ax^3 + bx^2 + cx + d$

$f'(x) = 4x^3 + 3ax^2 + 2bx + c$

$f''(x) = 12x^2 + 6ax + 2b$

Let, α is the root.

$\therefore \alpha^4 + a\alpha^3 + b\alpha^2 + c\alpha + d = 0$

$4\alpha^3 + 3a\alpha^2 + 2b\alpha + c = 0$

$12\alpha^2 + 6a\alpha + 2b = 0$

or, $6\alpha^2 + 3a\alpha + b = 0$

or, $b = -3a\alpha - 6\alpha^2$

$\therefore 4\alpha^3 + 3a\alpha^2 + 2\alpha(-3a\alpha - 6\alpha^2) + c = 0$

or, $4\alpha^3 + 3a\alpha^2 - 6a\alpha^2 - 12\alpha^3 + c = 0$

or, $c = 8\alpha^3 + 3a\alpha^2$

$\therefore \alpha^4 + a\alpha^3 + \alpha^2(-3a\alpha - 6\alpha^2) + \alpha(8\alpha^3 + 3a\alpha^2) + d = 0$

or, $\alpha^4 + a\alpha^3 - 3a\alpha^3 - 6\alpha^4 + 8\alpha^4 + 3a\alpha^3 + d = 0$

or, $d = -3\alpha^4 - a\alpha^3$

$$\therefore \frac{6c - ab}{3a^2 - 8b} = \frac{6(8\alpha^3 + 3a\alpha^2) - a(-3a\alpha - 6\alpha^2)}{3a^2 - 8(-3a\alpha - 6\alpha^2)}$$

$$= \frac{48\alpha^3 + 18a\alpha^2 + 3a^2\alpha + 6a\alpha^2}{3a^2 + 24a\alpha + 48\alpha^2} = \frac{48\alpha^3 + 3a^2\alpha + 24a\alpha^2}{3a^2 + 24a\alpha + 48\alpha^2}$$

$$= \frac{16\alpha^3 + a^2\alpha + 8a\alpha^2}{a^2 + 8a\alpha + 16\alpha^2} = \frac{\alpha(a^2 + 8a\alpha + 16\alpha^2)}{a^2 + 8a\alpha + 16\alpha^2} = \alpha$$

$\therefore$ one root of the equation $= \frac{6c - ab}{3a^2 - 8b}$ (Proved)

23) Let, $f(x) = x^4 + P_1x^3 + P_2x^2 + P_3x + P_4$ —(i)

Putting $x = i$ and $-i$ respectively into (i),

$f(i) = 1 - P_1 i - P_2 + iP_3 + P_4$ —(ii)

and $f(-i) = 1 + P_1 i - P_2 - iP_3 + P_4$ —(iii)

(ii) + (iii), $f(i) + f(-i) = 2(1 - P_2 + P_4)$

$(i-\alpha_1)(i-\alpha_2)(i-\alpha_3)(i-\alpha_4) + (-i-\alpha_1)(-i-\alpha_2)(-i-\alpha_3)(-i-\alpha_4) = 2(1 - P_2 + P_4)$

or,

1) Solve by Cardon's method $x^3+9x^2+15x-25=0$

⇒ The given equation,

$$x^3+9x^2+15x-25=0 \quad \text{---(i)}$$

Here $a_0=1$, $a_1=9$ and $n=3$

Let, the roots of the equation (i) be diminished by h to remove the 2nd terms.

$$\therefore h = -\frac{a_1}{a_0 n} = -\frac{9}{1\cdot 3} = -3$$

-3	1	9	15	-25
		-3	-18	9
	1	6	-3	-16
		-3	-9	
	1	3	-12	
		-3		
	1	0		
	1			

∴ The transfer equation is $z^3-12z+16=0$ ___(ii)

Let us solve (ii) by Cardon's method.

~~x^3+9x^2~~

Here, $H=-4$ and $G=-16$

$$\therefore G^2+4H^3 = 256+4(-4)^3 = 0$$

Let, $z=u+v$

and $uv=-H=4$ ___(iii)

$$\therefore u^3 = \frac{-G+\sqrt{G^2+4H^3}}{2} = \frac{16}{2} = 8$$

or, $u = 2, 2\omega, 2\omega^2$

$$\therefore v = \frac{4}{u} = \frac{4}{2}, \frac{4}{2\omega}, \frac{4}{2\omega^2} = 2, \frac{2}{\omega}, \frac{2}{\omega^2} = 2, 2\omega^2, 2\omega$$

$$\therefore z = 4,\ 2\omega+2\omega^2,\ 2\omega^2+2\omega = 4, -2, -2$$

∴ The roots of the given equation, $x = z+h$

$$= 4+(-3), (-2)+(-3), (-2)+(-3)$$

$$= 1, -5, -5$$

2) Solve the following equation by Cardon's method (i) $x^3-30x+133=0$

(ii) $x^3+3x+1=0$ (iii) $x^3-6x^2+30x-25=0$, (iv) $x^3-3x^2+12x+16=0$

ye.

Solution of bi quadratic equation: Ferrari's Method

1) Solve the following biquadratic equation by Ferrari's Method.

$$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$$

⇒ The given equation is $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ —(i)

The equation (i) can be written as

$$x^4 - 2x^3 = 5x^2 - 10x + 3$$

$$\text{or, } (x^2)^2 - 2\cdot x^2\cdot x + x^2 = 5x^2 - 10x + 3 + x^2$$

$$\text{or, } (x^2 - x)^2 = 6x^2 - 10x + 3 \quad \text{—(ii)}$$

$$\therefore \left(x^2 - x + \tfrac{1}{2}\lambda\right)^2 = 6x^2 - 10x + 3 + \cancel{2}\cdot(x^2 - x)\cdot\frac{1}{\cancel{2}}\lambda + \frac{1}{4}\lambda^2$$

$$= (6+\lambda)x^2 - (10+\lambda)x + \frac{1}{4}(\lambda^2 + 12) \quad \text{—(iii)}$$

Let, us now choose λ such that the R.H.S. of (iii) is a Perfect square.

$$\therefore (10+\lambda)^2 - 4(6+\lambda)\frac{1}{4}(\lambda^2+12) = 0$$

$$\text{or, } (\lambda^2+12)(\lambda+6) - (10+\lambda)^2 = 0$$

$$\text{or, } \lambda^3 + (6-1)\lambda^2 + (12-20)\lambda + (72-100) = 0$$

$$\text{or, } \lambda^3 + 5\lambda^2 - 8\lambda - 28 = 0 \quad \text{—(iv)}$$

$\lambda = -2$ is a root of the equation (iv).

Putting the value of $\lambda = -2$ into both sides of (iii) we have,

$$(x^2 - x - 1)^2 = 4x^2 - 8x + 4 = 4(x^2 - 2x + 1) = 4(x-1)^2$$

$$\therefore x^2 - x - 1 = \pm 2(x-1) \quad \text{—(v)}$$

taking positive sign in (v) we have,

$$x^2 - x - 1 = 2(x-1)$$

$$\text{or, } x^2 - 3x + 1 = 0$$

$$\text{or, } x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

taking negative sign in (v) we have,

$$x^2 - x - 2 = 2(1-x)$$

$$\text{or, } x^2 + x - 3 = 0$$

$$\text{or, } x = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

∴ The roots of the equation (i) are $\frac{3 \pm \sqrt{5}}{2}$ and $\frac{-1 \pm \sqrt{13}}{2}$.

2) Solve the following equation $x^4 - 18x^2 + 32x - 15 = 0$.

$\Rightarrow$ The given equation is $x^4 - 18x^2 + 32x - 15 = 0$ —— (i)

The equation (i) can be written as,

$x^4 = 18x^2 - 32x + 15$ —— (ii)

from (ii) $\left(x^2 + \frac{1}{2}\lambda\right)^2 = 18x^2 - 32x + 15 + x^2\lambda + \frac{1}{4}\lambda^2$

$= (18+\lambda)x^2 - 32x + \frac{1}{4}(\lambda^2 + 60)$ —— (iii)

Let us now choose λ such that the R.H.S. of (iii) is a perfect square.

$\therefore (-32)^2 - 4(18+\lambda)\cdot\frac{1}{4}(\lambda^2+60) = 0$

or, $(\lambda^2+60)(\lambda+18) - 1024 = 0$

~~$-1+18-60+56$~~

or, $\lambda^3 + 18\lambda^2 + 60\lambda + (60\times18 - 1024) = 0$.

or, $\lambda^3 + 18\lambda^2 + 60\lambda + 56 = 0$ —— (iv)

$\lambda = -2$ is a root of the equation (iv)

Putting this value of λ into (iii) we have,

$(x^2-1)^2 = 16x^2 - 32x + 16$

$= 16(x^2 - 2x + 1) = 16(x-1)^2$

or, $(x^2 - 1) = \pm 4(x-1)$

taking +ve sign,

$x^2 - 1 = 4x - 4$

or, $x^2 - 4x + 3 = 0$

or, $x = \frac{4 \pm \sqrt{16-12}}{2}$

$= \frac{4 \pm 2}{2} = 2 \pm 1$

$= 1, 3$

taking −ve sign,

$x^2 - 1 = 4 - 4x$

or, $x^2 + 4x - 5 = 0$

or, $x = \frac{-4 \pm \sqrt{16+20}}{2}$

$= \frac{-4 \pm \sqrt{36}}{2} = \frac{-4 \pm 6}{2}$

$= 1, -5$

$\therefore$ The roots of the equation (i) are $1, 3, 1, -5$

3) Solve the following biquadratic equation by Ferrari's method:—

i) $x^4 + 12x = 5$, ii) $x^4 + 3x^3 + x^2 - 2 = 0$, iii) $x^4 - 9x^3 + 28x^2 - 38x + 24 = 0$

~~Decarts~~ Descartes's Rule of Sign:-

An equation $f(x)=0$ with real cofficients can't have more positive roots than there are changes of sign in $f(x)$ and can't have more negative real roots than there are changes of sign in $f(-x)$. If the number of real roots be less the number of changes of sign then it will be by an even number.

1) Investigate the nature of the roots of the equation $x^6+x^4+x^2+x+3=0$ by Descarte's rule of sign.

⇒ Let, $f(x) = x^6+x^4+x^2+x+3$

∵ the number of changes of sign in $f(x)$ is 0, the equation has no positive real root.

Now, $f(-x) = x^6+x^4+x^2-x+3$

∵ the number of changes of sign in $f(-x)$ is 2, the equation has ~~max~~ atmost 2 negative real roots.

∵ the degree of the given equation is 6, the equation has 6 roots.

The maximum number of the real root of the equation is $0+2=2$

The maximum number of imaginary root is $6-2=4$

∴ The minimum number of imaginary root

2) Show that the equation $2x^7-x^4+4x^3-5=0$ has atleast 4 imaginary roots

⇒ Let, $f(x) = 2x^7-x^4+4x^3-5$.

The number of changes of sign in $f(x)$ is 3.

The number of changes has atmost 3 positive real roots.

∴ The given equation $-2x^7-x^4-4x^3-5$

Now, $f(-x) = -2x^7-x^4-4x^3-5$

∵ The number of changes of sign in $f(-x)$ is 0, the equation has no negetive real root,

∵ the degree of the given equation is 7, the equation has 7 roots.

∴ The maximum number of real roots of the equation is $3+0=3$

∴ The minimum number of imaginary roots is $7-3=4$ (Proved)

∴ The minimum number

Relation between roots and coefficients

Consider the equation

$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_{n-1}x+a_n=0$ —(i), $(a_0\neq 0)$

Let, $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation (i).

Then, $\sum\alpha_1 = -\frac{a_1}{a_0}$

$\sum\alpha_1\alpha_2 = \frac{a_2}{a_0}$

$$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3\alpha_4 = \frac{a_4}{a_0}$$

$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

1) Solve the equation $x^3-6x^2+3x+10=0$, given that the roots are in A.P.

→ the given equation, $x^3-6x^2+3x+10=0$ —(i)

Let, $(\alpha-\beta)$, α and $(\alpha+\beta)$ are the roots of the equation (i).

$\therefore (\alpha-\beta)+\alpha+(\alpha+\beta) = -\frac{-6}{1} = 6$ —(ii)

$\therefore (\alpha-\beta)\cdot\alpha+(\alpha-\beta)(\alpha+\beta)+\alpha(\alpha+\beta) = 3$ —(iii)

and $(\alpha-\beta)\alpha(\alpha+\beta) = -10$ —(iv)

$\therefore$ from (ii),

$3\alpha = 6$

or, $\alpha = 2$

$\therefore$ from (iv),

$\alpha(\alpha^2-\beta^2) = -10$

or, $2(4-\beta^2) = -10$

or, $\beta = \pm 3$

taking $\beta = 3$, the roots of the equation (i) are, $-1, 2, 5$

Home Work :-

Cardon's method's application :-

1) The given equation, $x^3-30x+133=0$ —(i)

Comparing equation (i) with standard cubic equation we get,

$3H = -30$ and $G = 133$

or, $H = -10$

Now, $G^2+4H^3 = (133)^2+4(-10)^3 = 17689-4000 = 13689 > 0$

$\therefore$ Cardon's method can be applied.

$\therefore$ Let, $x = u+v$

We have, $uv = -H = 10$

$\therefore u^3 = \frac{-G+\sqrt{G^2+4H^3}}{2} = \frac{-133+117}{2} = -8$

$\therefore u = -2, -2\omega, -2\omega^2$

$\therefore v = \frac{10}{u} = \frac{10}{-2}, \frac{10}{-2\omega}, \frac{10}{-2\omega^2} = -5, -5\omega^2, -5\omega$

$\therefore$ The ~~solut~~ roots of the equation (i) are, $x = u + v$

$= (-7), (-2\omega - 5\omega^2), (-2\omega^2 - 5\omega)$

ii) The given equation, $x^3 + 3x + 1 = 0$ ——(i)

Campairing (i) with standard cubic equation we get,

$3H = 3$ and $G = 1$

or, $H = 1$

Now, $G^2 + 4H^3 = 1 + 4 = 5 > 0$

$\therefore$ Cardon's method can be applied.

Let, $x = u + v$

we have, $uv = -H = -1$

$\therefore u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2} = \frac{-1 + \sqrt{5}}{2} = 0.6$

or, $u = 0.85, 0.85\omega, 0.85\omega^2$

$\therefore v = \frac{-1}{u} = \frac{-1}{0.85}, \frac{-1}{0.85\omega}, \frac{-1}{0.85\omega^2} = -1.17, -1.17\omega^2, -1.17\omega$

$\therefore$ The roots of the equation, $x = u + v = (0.85 - 1.17), (0.85\omega - 1.17\omega^2), (0.85\omega^2 - 1.17\omega)$

$= -0.32, (0.85\omega - 1.17\omega^2), (0.85\omega^2 - 1.17\omega)$

iii) The given equation,

$x^3 - 6x^2 + 30x - 25 = 0$ ——(i)

Here, $a_0 = 1$, $a_1 = -6$, $n = 3$

The roots of the equation (i) diminished by h to remove the second term, where $h = -\frac{a_1}{a_0 n} = \frac{+6}{3 \times 1} = 2$

$$\begin{array}{c|cccc} 2 & 1 & -6 & 30 & -25 \\ & & 2 & -8 & 44 \\ \hline & 1 & -4 & 22 & \boxed{19} \\ & & 2 & -4 & \\ \hline & 1 & -2 & \boxed{18} & \\ & & 2 & & \\ \hline & 1 & \boxed{0} & & \\ \hline & \boxed{1} & & & \end{array}$$

$\therefore$ The transform equation is $x^3 + 18x + 19 = 0$ ——(ii)

Let us, solve (ii) by Cardon's method.

Here, $3H = 18$ and $G = 19$

or, $H = 6$

Now, $G^2 + 4H^3 = 361 + 864 = 1225 > 0$

$\therefore$ Cardon's method can be applied.

Let, $z = u + v$

We have,

$uv = -H = -6$

$\therefore u^3 = \dfrac{-G + \sqrt{G^2 + 4H^3}}{2} = -\dfrac{19 + 35}{2} = 8$

or, $u = 2, 2\omega, 2\omega^2$

$\therefore v = \dfrac{-6}{u} = \dfrac{-6}{2}, \dfrac{-6}{2\omega}, \dfrac{-6}{2\omega^2} = -3, -3\omega^2, -3\omega$

$\therefore z = u + v = -1, 2\omega - 3\omega^2, 2\omega^2 - 3\omega$

$\therefore$ The roots of the equation (i), $x = z + h$

$= -1+2, 2\omega - 3\omega^2 + 2, 2\omega^2 - 3\omega + 2$

$\cancel{= 1, 2\omega - 3\omega^2}$

$= 1, -2\omega^2 - 3\omega^2, -2\omega - 3\omega$

$= 1, -5\omega^2, -5\omega$

iv) The given equation, $x^3 - 3x^2 + 12x + 16 = 0$ ___ (i)

Here, $a_0 = 1$, $a_1 = -3$ and $n = 3$

The roots of the equation (i) ~~can~~ be diminished by h to remove second term,

where, $h = -\dfrac{a_1}{a_0 n} = +\dfrac{3}{3} = 1$

$$\therefore \begin{array}{c|cccc} 1 & 1 & -3 & 12 & 16 \\ & & 1 & -2 & 10 \\ \hline & 1 & -2 & 10 & 6 \\ & & 1 & -1 & \\ \hline & 1 & -1 & 9 & \\ & & 1 & & \\ \hline & 1 & 0 & & \\ & & & & \\ \hline & 1 & & & \end{array}$$

$\therefore$ The transformed equation is

$z^3 + 9z + 6 = 0$ ___ (ii)

Let us, solve (ii) by Cardon's method.

Here, $3H = 9$ and $G = 6$

or, $H = 3$

Now, $G^2 + 4H^3 = 36 + 108 = 144 > 0$

$\therefore$ Cardon's method can be applied.

Let, $z = u + v$

We have, $uv = -H = -3$

$\therefore u^3 = \dfrac{-G + \sqrt{G^2 + 4H^3}}{2} = \dfrac{-6 + 12}{2} = 3$

or, $u = 1.4, 1.4\omega, 1.4\omega^2$

$\therefore v = \dfrac{-3}{1.4}, \dfrac{-3}{1.4\omega}, \dfrac{-3}{1.4\omega^2} = -2.14, -2.14\omega^2, -2.14\omega$

$\therefore z = u + v = -0.74, 1.4\omega - 2.14\omega^2, 1.4\omega^2 - 2.14\omega$

$\therefore$ The roots of the equation (i), $x = z + h$

$= 2.26, 1.4\omega - 2.14\omega^2 + 3, 1.4\omega^2 - 2.14\omega + 3$

Application of Ferrari's method :—

i) The given equation, $x^4+12x=5$ —(i)

The given equation can be written as,

$(x^2)^2 = -12x+5$ —(ii)

from (i) $\left(x^2+\frac{1}{2}\lambda\right)^2 = -12x+5+x^2\lambda+\frac{1}{4}\lambda^2$

$= x^2\lambda - 12x + \frac{1}{4}(\lambda^2+20)$ —(iii)

Let us choose λ such that the R.H.S. of (iii) can be perfect square,

$\therefore 144-\lambda(\lambda^2+20)=0$

or, $144-\lambda^3-20\lambda=0$

or, $\lambda^3+20\lambda-144=0$

$\therefore \lambda=4$

$\therefore$ putting $\lambda=4$ into (iii),

$(x^2+2)^2 = 4x^2-12x+\frac{36}{4} = 4x^2-12x+9 = (2x-3)^2$

or, $x^2+2 = \pm(2x-3)$

taking +ve sign,

$x^2+2=2x-3$

or, $x^2-2x+5=0$

or, $x=\frac{2\pm\sqrt{4-20}}{2}$

$=\frac{2\pm i4}{2} = 1\pm 2i$

taking −ve sign,

$x^2+2=-2x+3$

or, $x^2-2x-1=0$

or, $x=\frac{2\pm\sqrt{4+4}}{2}$

$=\frac{2\pm 2\sqrt{2}}{2} = 1\pm\sqrt{2}$

$\therefore$ The roots are $(1\pm 2i)$ and $(1\pm\sqrt{2})$

ii) The given equation, $x^4+3x^3+x^2-2=0$ —(i)

The given equation can be written as,

$x^4+3x^3 = 2-x^2$

or, $(x^2)^2+2\cdot\frac{3}{2}x\cdot x^2+\frac{9}{4}x^2 = 2-x^2+\frac{9}{4}x^2$

or, $\left(x^2+\frac{3}{2}x\right)^2 = \frac{5}{4}x^2+2$ —(ii)

$\frac{9}{4}-1 = \frac{5}{4}$

from (ii) $\left(x^2+\frac{3}{2}x+\frac{\lambda}{2}\right)^2 = \frac{5}{4}x^2+2+\frac{\lambda^2}{4}+\lambda\left(x^2+\frac{3}{2}x\right)$

$=\left(\frac{5}{4}+\lambda\right)x^2+\frac{3}{2}\lambda x+\frac{1}{4}(\lambda^2+8)$ —(iii)

Let us choose λ such that R.H.S. of (iii) can be perfect square,

$\frac{9}{4}\lambda^2-(\lambda^2+8)\left(\frac{5}{4}+\lambda\right)=0$

or, $\frac{9}{4}\lambda^2 - \frac{5}{4}\lambda^2 - \lambda^3 - 10 - 8\lambda = 0$

or, $9\lambda^2 - 5\lambda^2 - 4\lambda^3 - 40 - 32\lambda = 0$

or, $4\lambda^3 - 4\lambda^2 + 32\lambda + 40 = 0$

or, $\lambda^3 - \lambda^2 + 8\lambda + 10 = 0$

$\therefore \lambda = -1$

Putting $\lambda = -1$ into (iii),

$$\left(x^2 + \frac{3}{2}x - \frac{1}{2}\right)^2 = \left(\frac{5}{4} - 1\right)x^2 - \frac{3}{2}x + \frac{9}{4}$$

$$= \frac{1}{4}x^2 - \frac{3}{2}x + \left(\frac{3}{2}\right)^2$$

$$= \left(\frac{1}{2}x\right)^2 - 2\cdot\frac{1}{2}x\cdot\frac{3}{2} + \frac{9}{4}$$

$$= \left(\frac{1}{2}x - \frac{3}{2}\right)^2$$

1) ⇒

or, $x^2 + \frac{3}{2}x - \frac{1}{2} = \pm\left(\frac{1}{2}x - \frac{3}{2}\right)$

taking +ve sign,

$x^2 + \frac{3}{2}x - \frac{1}{2} = \frac{1}{2}x - \frac{3}{2}$

or, $x^2 + x + 1 = 0$

or, $x = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm i\sqrt{3}}{2}$

taking -ve sign, $\frac{3}{2} - \frac{1}{2}$

$x^2 + \frac{3}{2}x - \frac{1}{2} = \frac{3}{2} - \frac{1}{2}x$

or, $x^2 + 2x - 2 = 0$

$\therefore x = \frac{-2 \pm \sqrt{4+8}}{2}$

$= \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$

$\therefore$ The solution $x = \frac{-1 \pm i\sqrt{3}}{2}$ and $x = -1 \pm \sqrt{3}$.

iii) The given equation,

$x^4 - 9x^3 + 28x^2 - 38x + 24 = 0$ —(i)

$\frac{81}{4} - 28$

$\frac{81 - 112}{4}$

The given equation can be written as

$x^4 - 9x^3 = 38x - 28x^2 - 24$

or, $(x^2)^2 - 2\cdot x^2 \cdot \frac{9}{2}x + \frac{81}{4}x^2 = 38x - 28x^2 - 24 + \frac{81}{4}x^2$

or, $\left(x^2 - \frac{9}{2}x\right)^2 = -\frac{31}{4}x^2 + 38x - 24$ —(ii)

2)

from (ii), $\left(x^2 - \frac{9}{2}x + \frac{\lambda}{2}\right)^2 = -\frac{31x^2}{4} + 38x - 24 + \frac{\lambda^2}{4} + \lambda\left(x^2 - \frac{9}{2}x\right)$

$= \left(\lambda - \frac{31}{4}\right)x^2 + \left(38 - \frac{9}{2}\lambda\right)x + \frac{1}{4}(\lambda^2 - 96)$ —(iii)

⇒

Let us choose λ such that R.H.S. of (iii) can be perfectly square

$\therefore \left(38-\frac{9}{2}x\right)^2 - (x^2-96)\left(x-\frac{3}{4}\right)=0$

or, $1444+\frac{81}{4}x^2-342x-x^3+\frac{3}{4}x^2+96x-72=0$

or, $x^3-21x^2+246x-1372=0$

---

1) If α be a multiple root of order 3 of the equation $x^4+bx^2+cx+d=0$ then show that $\alpha = -\frac{8d}{3c}$.

$\Rightarrow$ The given equation, $x^4+bx^2+cx+d=0$ — (i)

Let, α, α, α and β be the roots of the equation (i).

$\therefore$ from relation between roots and the coefficients we have,

$\sum\alpha = \alpha+\alpha+\alpha+\beta = 0 \Rightarrow 3\alpha+\beta=0$ — (ii)

$\sum\alpha\beta = \alpha\alpha+\alpha\alpha+\alpha\beta+\alpha\alpha+\alpha\beta+\alpha\beta = b \Rightarrow 3\alpha^2+3\alpha\beta=b$ — (iii)

$\sum\alpha\beta\gamma = \alpha^3+\alpha^2\beta+\alpha^2\beta+\alpha^2\beta=-c \Rightarrow \alpha^3+3\alpha^2\beta=-c$ — (iv)

$\therefore \alpha^3\beta = d$ — (v)

From (ii), $\beta=-3\alpha$

Putting this value of β into (iv) and (v) we have,

$\alpha^3+3\alpha^2(-3\alpha)=-c$ and $\alpha^3(-3\alpha)=d$

or, $\alpha^3=\frac{c}{8}$ — (vi) $\qquad \alpha^4=-\frac{d}{3}$ — (vii)

$\therefore$ from (vi) and (vii), $\alpha=\frac{\alpha^4}{\alpha^3}=\frac{-d\,8}{3c}=-\frac{8d}{3c}$ (Proved)

2) If one of the roots of the equation $x^3+ax+b=0$ is twice the difference of the other two, show that one root of the equation is $\frac{13b}{3a}$.

$\Rightarrow$ The given equation is, $x^3+ax+b=0$

Let, α, β, γ be the roots of the equation (i).

$\therefore$ By the given condition, $\alpha=2(\beta-\gamma)$ — (ii)

$\Rightarrow$ again from relation between roots and the coefficients we have,

$\alpha+\beta+\gamma=0$

or, $\alpha=-\beta-\gamma$ — (iii)

from (ii) and (iii)

$-\beta-\gamma=2\beta-2\gamma$

or, $3\beta=\gamma$

or, $\beta=\frac{\gamma}{3}$ — (iv)

from (iv) and (iii),

$\alpha=-\frac{4}{3}\gamma$

$\therefore \gamma=-\frac{3}{4}\alpha$ — (v)

$\because \gamma$ is a root of the equation (i),

$\therefore \gamma^3+a\gamma+b=0$

or, $\left(-\frac{3}{4}\alpha\right)^3+a\left(-\frac{3}{4}\alpha\right)+b=0$

or, $27\alpha^3+48a\alpha-64b=0$ — (vi)

again, since α is a root of the equation (i),

$\therefore \alpha^3+a\alpha+b=0$ — (vii)

eliminating α^3 between (vi) and (vii),

$\cancel{27\alpha^3}+48a\alpha-64b=0$

$-\ \cancel{\alpha^3}+a\alpha+b=0 \quad \times 27$

$21a\alpha-91b=0$

or, $\alpha=\frac{91b}{21a}=\frac{13b}{3a}$

3) If the product of two roots of the equation $x^4+px^3+qx^2+rx+s=0$ equal to the product of the other two, prove that $r^2=p^2s$.

$\Rightarrow$ the given equation,

$x^4+px^3+qx^2+rx+s=0$ — (i)

Let,

$\alpha,\beta,\gamma,\delta$ be the roots of the equation (i).

By the given condition,

$\alpha\beta=\gamma\delta$ — (ii)

Now from the relation between roots and the coefficient,

$\alpha+\beta+\gamma+\delta=-p$ — (iii)

$\alpha\beta\gamma\delta=s$ — (iv)

from (ii) and (iv),

$\alpha\beta=\gamma\delta=\sqrt{s}$ — (v) also, $\Sigma \alpha\beta = q$

or, $\alpha\beta + \alpha\gamma + \gamma\delta + \beta\gamma + \beta\delta + \gamma\delta = q$

or, $\sqrt{s} + \alpha\gamma + \gamma\delta + \beta\gamma + \beta\delta + \sqrt{s} = q$

or, $(\alpha+\beta)(\gamma+\delta) = q - 2\sqrt{s}$ —— (vi)

We have,

$\Sigma \alpha\beta\gamma = -r$

or, $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$

or, $\sqrt{s}\gamma + \sqrt{s}\delta + \alpha\sqrt{s} = \beta\sqrt{s} = -r$

or, $\sqrt{s}(\gamma+\beta+\delta+\gamma) = -r$

or, $\sqrt{s}(-p) = -r$

or, $r^2 = p^2 s$ (Proved)

4) If α, β, γ be the roots of the equation $ax^3 + bx^2 + c = 0$, $a \neq$ then find the value of $\Sigma \alpha^2$.

⇒ the given equation, $ax^3 + bx^2 + c = 0$ —— (i)

∵ α, β, γ are the roots of the equation (i), from relation between roots and the coefficients we have,

$\Sigma \alpha = -\frac{b}{a}$ —— (ii)

$\Sigma \alpha\beta = 0$ —— (iii)

$\alpha\beta\gamma = -\frac{c}{a}$ —— (iv)

∴ $\Sigma \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (\Sigma \alpha)^2 - 2\Sigma\alpha\beta = \left(-\frac{b}{a}\right)^2 - 2 \cdot 0$

$= \frac{b^2}{a^2}$

5) If α, β, γ be the roots of the equation $x^3 - 3x + 1 = 0$, find the equation whose roots are $\alpha^2 + \beta^2 - \gamma^2$, $\alpha^2 + \gamma^2 - \beta^2$, $\beta^2 + \gamma^2 - \alpha^2$.

⇒ The given equation, $x^3 - 3x + 1 = 0$ —— (i)

From relation between roots and the coefficients,

$\Sigma \alpha = 0$ —— (ii), $\Sigma \alpha\beta = -3$ —— (iii), $\alpha\beta\gamma = -1$ —— (iv)

Let, $y = \beta^2 + \gamma^2 - \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha^2 = \Sigma\alpha^2 - 2\alpha^2$

$= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta - 2\alpha^2 = 0 - 2(-3) - 2\alpha^2$

$= 6 - 2\alpha^2$

∴ $\alpha^2 = \frac{6-y}{2}$

$\because \alpha$ is a root of the equation (i),

$\therefore \alpha^3 - 3\alpha + 1 = 0$

or, $\alpha(\alpha^2 - 3) = -1$

or, $\alpha^2(\alpha^2 - 3)^2 = 1$

or, $\left(\frac{6-y}{2}\right)\left\{\frac{6-y}{2} - 3\right\}^2 = 1$

or, $\frac{6-y}{2} \times \left\{\frac{(6-y)^2}{4} + 9 - 18 + 3y\right\} = 1$

or, $\frac{6-y}{2} \times \left\{\frac{\cancel{36} + y^2 - \cancel{12y} + \cancel{12y} - \cancel{36}}{4}\right\} = 1$

or, $(6-y)(y^2) = 8$

or, $6y^2 - y^3 = 8$

or, $y^3 - 6y^2 + 8 = 0$

$\therefore$ This is the required equation.

6) Find the equation whose roots are cube of the roots of the equation $x^3 + 3x^2 + 2 = 0$

$\Rightarrow$ The given equation $x^3 + 3x^2 + 2 = 0$ —— (i)

Let, α, β, γ be the roots of the equation (i),

We ~~are~~ have to construct a equation whose roots are $\alpha^3, \beta^3, \gamma^3$.

Let, $y = \alpha^3$

$\because \alpha$ is a root of the equation (i)

$\therefore \alpha^3 + 3\alpha^2 + 2 = 0$

or, $(\alpha^3 + 2)^3 = (-3\alpha^2)^3$

or, $(\alpha^3 + 2)^3 = -27(\alpha^3)^2$

or, $(y+2)^3 = -27y^2$

or, $y^3 + 8 + 6y^2 + 12y = -27y^2$

or, $y^3 + 33y^2 + 12y + 8 = 0$

7) Find the equation whose roots are square of the difference of the roots of the equation $x^3+3x+1=0$

$\Rightarrow$ The given equation, $x^3+3x+1=0$ —(i)

Let, α, β, γ be the roots of the equation (i).

$\therefore$ from relation between roots and the coefficients,

$\sum\alpha = 0$ —(ii), $\sum\alpha\beta = 3$ —(iii); $\alpha\beta\gamma = -1$ —(iv)

We have to construct an equation whose roots are $(\alpha-\beta)^2$, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$.

Let, $y = (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = (-\gamma)^2 - 4\cdot\frac{\alpha\beta\gamma}{\gamma}$

$= \gamma^2 - 4\cdot\frac{-1}{\gamma} = \frac{\gamma^3+4}{\gamma}$

or, $\gamma^3 - y\gamma + 4 = 0$ —(v)

again since γ is a root of the equation (i),

$\therefore$ $\gamma^3 + 3\gamma + 1 = 0$ —(vi)

from (v) and (vi),

$(y+3)\gamma = 3$

or, $\gamma = \frac{3}{y+3}$

$\therefore$ $\left(\frac{3}{y+3}\right)^3 + 3\cdot\frac{3}{y+3} + 1 = 0$

or, $\frac{27}{(y+3)^3} + \frac{9}{(y+3)} + 1 = 0$

or, $27 + 9(y+3)^2 + (y+3)^3 = 0$

or, $27 + 9y^2 + 81 + 54y + 27 = 0$

or, $y^3 + 27 + 9y^2 + 27y + 9y^2 + 81 + 54y + 27 = 0$

or, $y^3 + 18y^2 + 81y + 135 = 0$

8) If α, β are any two roots of the equation $x^3+qx+r=0$, formed the equation whose roots are the 6 values $\frac{\alpha}{\beta}$.

$\Rightarrow$ The given equation, $x^3+qx+r=0$ —(i)

Let, $y = \frac{\alpha}{\beta}$

$\therefore \alpha = y\beta$

$\because \alpha, \beta$ be the roots of the equation (i)

$\therefore$ $\alpha^3 + q\alpha + r = 0$ —(ii)

$\beta^3 + q\beta + r = 0$ —(iii)

from (ii) we have,

$$~~\beta^3 + y\beta + r = 0~~ \quad \text{(iii)}$$

$$y^3\beta^3 + qy\beta + r = 0 \quad \text{(iv)}$$

from (iii) and (iv),

$$q(y^3 - y)\beta + r(y^3 - 1) = 0$$

$$\text{or, } \beta = \frac{r(y^3-1)}{q(y-y^3)}$$

Putting the value of β into (iii),

$$\left(\frac{r(y^3-1)}{q(y-y^3)}\right)^3 + \cancel{q} \cdot \frac{r(y^3-1)}{\cancel{q}(y-y^3)} + r = 0$$

9) If α, β, γ be the roots of the equation $x^3 - 3px^2 + 3(p-1)x + 1 = 0$ then find the equation whose roots are $1-\alpha, 1-\beta, 1-\gamma$.

$\Rightarrow$ $x^3 - 3px^2 + 3(p-1)x + 1 = 0$ — (i)

Let, $y = 1 - \alpha$

or, $\alpha = 1 - y$

$\because$ α be the root of (i)

$\alpha^3 - 3p\alpha^2 + 3(p-1)\alpha + 1 = 0$

$\therefore$ $\alpha^3 - 3p\alpha^2 + 3(p-1)(1-y) + 1 = 0$

$\therefore (1-y)^3 - 3p(1-y)^2 + 3(p-1)(1-y) + 1 = 0$

$\therefore (1-y)^3 - 3p(1-y)^2 + 3(p-1)(1-y)$ ... $- 3p(1-y)^2 + 3(p-1)(1-y)$

$\therefore (1-y)^3 - 3p(1-y)^2 + 3(p-1) ... - 3py^2 + 6py + \cancel{3p} - 3py - 3 + \cancel{3y} + 1 = 0$

or, $1 - \cancel{3y} + 3y^2 - y^3 - \cancel{3p} - 3py^2 + 6py + \cancel{3p} - 3py - 3 + \cancel{3y} + 1 = 0$

or, $y^3 - 3y^2 + 3py^2 + 3py - 1 = 0$

<u>Some results :-</u>

1) $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

2) $\sum\alpha^3 = \sum\alpha^2 \sum\alpha - \sum\alpha^2\beta$

3) $\sum\alpha^2\beta = \sum\alpha\beta\sum\alpha - 3\sum\alpha\beta\gamma$

4) $\sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\alpha\beta\gamma\sum\alpha$

5) $\sum\alpha^3\beta^3 = (\sum\alpha\beta)^3 - 3\{\alpha\beta\gamma\sum\alpha\sum\alpha\beta - \alpha^2\beta^2\gamma^2\}$

6) $\sum\alpha^4 = (\sum\alpha^2)^2 - 2\sum\alpha^2\beta^2$

7) $\sum\alpha^3\beta = \sum\alpha^2\sum\alpha\beta$, 8) ~~$\sum\alpha^3\beta = \sum\alpha^2\sum\alpha\beta - \sum\alpha^2\beta\gamma$~~

9) $\sum\alpha^2\beta\gamma = \sum\alpha\beta\gamma\sum\alpha - 4\alpha\beta\gamma\delta$

10) If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ then find the value of $\sum \frac{1}{\alpha^2}$, $\sum \alpha^2\beta$

⇒ $x^3+px^2+qx+r=0$ ——— (i)

∵ α, β, γ are the root of (i)

from the relation between roots and coefficient,

$\sum\alpha = -p$, $\sum\alpha\beta = q$, $\alpha\beta\gamma = -r$ ——— (ii)

Now, $\sum\frac{1}{\alpha^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\beta^2}{\alpha^2\beta^2\gamma^2} = \frac{\sum\alpha^2\beta^2}{(\alpha\beta\gamma)^2}$

$= \frac{(\sum\alpha\beta)^2 - 2\alpha\beta\gamma\sum\alpha}{(-r)^2}$

$= \frac{q^2+2\alpha\beta\gamma p}{r^2}$ (from (ii)) $= \frac{q^2+2rp}{r^2}$

$\therefore \sum\alpha^2\beta = \sum\alpha\beta\sum\alpha - 3\sum\alpha\beta\gamma$

$= -qp+3r$ (from (ii))

11) If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ then find the values of $\sum\alpha^3$ and $\sum\alpha^2\beta^2$,

⇒ $x^3+px^2+qx+r=0$ ——— (i)

∵ α, β, γ are the root of (i)

∴ from the relation between roots and coefficients,

$\sum\alpha = -p$, $\sum\alpha\beta = q$, $\alpha\beta\gamma = -r$ ——— (ii)

$\therefore \sum\alpha^3 = \sum\alpha^2\sum\alpha - \sum\alpha^2\beta$

$= \{(\sum\alpha)^2 - 2\sum\alpha\beta\}\sum\alpha - \{\sum\alpha\beta\sum\alpha - 3\sum\alpha\beta\gamma\}$

$= (p^2-2q)(-p) - (-qp+3r)$

$= -p^3+2pq+qp-3r = -p^3+3pq-3r$.

$\therefore \sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\alpha\beta\gamma\sum\alpha$

$= q^2 - \cancel{2\alpha\beta\gamma}\ 2(-r)(-p)$

$= q^2-2rp$

12) If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ then find the values of $\Sigma\frac{1}{\alpha}$, $\Sigma\frac{1}{\alpha\beta}$, $\Sigma\frac{1}{\alpha^2}$

→ $x^3+px^2+qx+r=0$ ——— (1)

$\because \alpha, \beta, \gamma$ are the roots of (1).

$\therefore \Sigma\alpha=-p, \quad \Sigma\alpha\beta=q, \quad \alpha\beta\gamma=-r$

$$\therefore \Sigma\frac{1}{\alpha} = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\beta+\gamma\alpha}{\alpha\beta\gamma} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{q}{-r} = -\frac{q}{r}$$

$$\therefore \Sigma\frac{1}{\alpha\beta} = \frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}$$

$$= \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma} = \frac{\Sigma\alpha}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$$

$$\therefore \Sigma\frac{1}{\alpha^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\beta^2\gamma^2+\alpha^2\gamma^2+\beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{\Sigma\alpha^2\beta^2}{(\alpha\beta\gamma)^2} = \frac{(\Sigma\alpha\beta)-2\alpha\beta\gamma\Sigma\alpha}{r^2}$$

$$= \frac{q^2+2\alpha\beta\gamma p}{r^2}$$

$$= \frac{q^2+2p(-r)}{r^2} = \frac{q^2-2pr}{r^2}$$

Theory of eqⁿ

1) If α be an imaginary root of the eqⁿ $x^n-1=0$, where n is a Prime number then the value of $(1-\alpha)(1-\alpha^2)(1-\alpha^3)\dots(1-\alpha^{n-1})$ is – i) 0, ii) n, iii) $n-1$, iv) none of these.

⇒ We have, $x^n-1=0$

or, $x^n=1=\cos 0+i\sin 0=\cos 2k\pi+i\sin 2k\pi$, $k\in\mathbb{Z}$

or, $x=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$, $k=0,1,2,\dots,n-1$

or, $x=\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^k$, $k=0,1,2,\dots,n-1$

$=\alpha^k$, $k=0,1,2,\dots,n-1$ $\left[\text{let, } \alpha=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right]$

$=1,\alpha,\alpha^2,\alpha^3,\dots,\alpha^{n-1}$

$\therefore x^n-1=(x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)\dots(x-\alpha^{n-1})$

$\lim_{x\to1}\frac{x^n-1}{x-1}=\lim_{x\to1}(x-\alpha)(x-\alpha^2)(x-\alpha^3)\dots(x-\alpha^{n-1})$

$\lim_{x\to1}\frac{nx^{n-1}}{1}=(1-\alpha)(1-\alpha^2)(1-\alpha^3)\dots(1-\alpha^{n-1})=n(1)^{n-1}=n$

2) The eqⁿ $x^3-3x^2-9x+27=0$ has – i) a multiple root ii) all roots real iii) both (i) and (ii), iv) neither (i) nor (ii)

⇒ Let, $f(x)=x^3-3x^2-9x+27$

$\therefore f'(x)=3x^2-6x-9$
$=3(x^2-2x-3)$

$\therefore$ We have,

$$\begin{array}{r|l}
x^2-2x-3 & x^3-3x^2-9x+27 \;\big|\; x-1 \\
 & \underline{x^3-2x^2-3x} \\
 & -x^2-6x+27 \\
 & \underline{-x^2+2x+3} \\
-8 & -8x+24 \\
x-3 & x^2-2x-3 \;\big|\; x+1 \\
 & \underline{x^2-3x} \\
 & x-3 \\
 & \underline{x-3} \\
 & 0
\end{array}$$

$\therefore$ 3 is a double root of $f(x)=0$.

$\therefore$ The given eqⁿ has all roots real.

3) Find the value of k for which the eqⁿ $x^3-9x^2+24x+k=0$ may have multiple root. Find also the roots of the eqⁿ.

⇒ Let, $f(x)=x^3-9x^2+24x+k$

$\therefore f'(x)=3x^2-18x+24=3(x^2-6x+8)=3(x-2)(x-4)$

Case-I:– if 2 be a multiple root then $f(2)=0$

$\therefore 8-36+48+k=0$

$k=-20$

in this case,

$x^3-9x^2+24x-20=0$

$\Rightarrow x^3-2x^2-7x^2+14x+10x-20=0$

$\Rightarrow x^2(x-2)-7x(x-2)+10(x-2)=0$

$\Rightarrow (x-2)(x^2-7x+10)=0$

$\Rightarrow x=2,\quad x^2-5x-2x+10=0$

$x(x-5)-2(x-5)=0$

$(x-5)(x-2)=0$

$x=5,2$

Case-II :- If 4 is a multiple root of the given eq^n then $f(4)=0$

$\therefore 64-144+96+k=0$

or, $k=-16$

$\therefore$ in this case, $x^3-9x^2+24x-16=0$

$\therefore x=4,4,1$

4) If $1,\alpha,\beta,\gamma,\delta$ are the roots of the eq^n $x^5-1=0$ then prove that $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)=5$

$\Rightarrow$ since, $1,\alpha,\beta,\gamma,\delta$ are the roots of the given eq^n

$\therefore x^5-1=(x-1)(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$\therefore \lim_{x\to1}\frac{x^5-1}{x-1}=\lim_{x\to1}(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

or, $5=(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)$ (proved).

5) Find the area of the triangle of which the lengths of the sides are the root the eq^n $x^3-ax^2+bx-c=0$

$\Rightarrow$ The given eq^n is; $x^3-ax^2+bx-c=0$ —(i)

Let, α,β,γ be the roots of the eq^n (i)

$\therefore$ from relation between roots and coefficients we have,

$\alpha+\beta+\gamma=a,\quad \alpha\beta+\beta\gamma+\alpha\gamma=b,\quad \alpha\beta\gamma=c$ —(ii)

The semiperimeter of the triangle is $s=\frac{\alpha+\beta+\gamma}{2}=\frac{a}{2}$

$\therefore$ The area of the triangle, $\Delta=\sqrt{s(s-\alpha)(s-\beta)(s-\gamma)}$

$=\sqrt{s\{s^3-(\alpha+\beta+\gamma)s^2+(\alpha\beta+\beta\gamma+\gamma\alpha)s-\alpha\beta\gamma\}}$

$=\sqrt{\frac{a}{2}\left\{\frac{a^3}{8}-as^2+bs-c\right\}}$

$=\sqrt{\frac{a}{2}\left\{\frac{a^3}{8}-\frac{a^3}{4}+\frac{ab}{2}-c\right\}}=\frac{\sqrt{a^4-2a^4+4a^2b-8ca}}{4}=\frac{\sqrt{4a^2b-8ac-a^4}}{4}$

$=\frac{\sqrt{a(4ab-8c-a^3)}}{4}$

6) If α, β, γ be the roots of the eqⁿ $x^3+x+1=0$ then Prove that $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)=1$

Since, α, β, γ are the roots of the eqⁿ $x^3+x+1=0$ — (i)

$\therefore x^3+x+1 = (x-\alpha)(x-\beta)(x-\gamma)$ — (ii)

Putting $x = i$ into both sides of (ii) we have,

$$(i-\alpha)(i-\beta)(i-\gamma) = 1 \text{ — (iii)}$$

Simmilarly, Putting $x=-i$ into both sides of (ii),

$$-(i+\alpha)(i+\beta)(i+\gamma) = 1 \text{ — (iv)}$$

multiplying (iii) and (iv) column wise,

$$-(-1-\alpha^2)(-1-\beta^2)(-1-\gamma^2) = 1$$

or, $(1+\alpha^2)(1+\beta^2)(1+\gamma^2) = 1$

7) Prove that x^2+x+1 is a factor of $x^{10}+x^5+1$.

⇒ We have, $x^2+x+1 = (x-\omega)(x-\omega^2)$

Let, $\phi(x) = x^{10}+x^5+1$

$\therefore \phi(\omega) = \omega^{10}+\omega^5+1 = \omega+\omega^2+1=0$ $\therefore (x-\omega)$ is a factor of $\phi(x)$

$\therefore \phi(\omega^2) = \omega^{20}+\omega^{10}+1 = \omega^2+\omega+1=0$

$\therefore (x-\omega^2)$ is a factor of $\phi(x)$

$\therefore x^2+x+1$ is a factor of $x^{10}+x^5+1$.

8) Obtain the condition that $x^3+3px+q$ has a factor $(x-a)^2$.

⇒ Let, $f(x) = x^3+3px+q$

$\therefore$ We are to find the condition that 'a' is a double root of the eqⁿ $f(x)=0$

$\therefore f(a)=0 \Rightarrow a^3+3pa+q=0$

and $f'(a)=0 \Rightarrow 3a^2+3p=0$

$\therefore a^2 = -p$

$a^3+3pa+q=0$

$a^2(a^2+3p)^2 = (-q)^2 = q^2$

$-p(-p+3p)^2 = q^2$

$-p(2p)^2 = q^2$

$\Rightarrow -4p^3 = q^2$

or, $4p^3+q^2=0$

9) Let $f(x)$ be a polynomial and $a \neq b$ be two real numbers. Show that that the remainder in the division of $f(x)$ by $(x-a)(x-b)$ is $\dfrac{(x-b)f(a)-(x-a)f(b)}{a-b}$

$\Rightarrow$ Let, $f(x) = (x-a)(x-b)Q(x) + Ax + B$ —— (i)

Putting $x = a$ and $x = b$ into (i) we have,

$f(a) = Aa + B$ —— (ii)

$f(b) = Ab + B$ —— (iii)

from (ii) and (iii)

$A = \dfrac{f(a) - f(b)}{a-b}$

$\therefore B = f(b) - \dfrac{f(a)b - f(b)b}{a-b} = \dfrac{af(b) - f(a)b}{a-b}$

$\therefore$ The remainder is $Ax + B = \dfrac{f(a)x - f(b)x}{a-b} + \dfrac{af(b) - f(a)b}{a-b}$

$= \dfrac{(x-b)f(a) - (x-a)f(b)}{a-b}$ (Proved)

10) If α, β, γ are the roots of the eq^n $x^3 + px^2 + qx + r = 0$ then find the value of $\Sigma \frac{1}{\alpha^2}$, and $\Sigma \alpha^2\beta$.

$\Rightarrow$ $x^3 + px^2 + qx + r = 0$ —— (i)

Since, α, β, γ are the roots of the eq^n (i).

$\therefore \Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\alpha\beta\gamma = -r$ —— (ii)

$\therefore \Sigma \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \dfrac{\Sigma\alpha^2\beta^2}{(\alpha\beta\gamma)^2} = \dfrac{(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\Sigma\alpha}{r^2}$

$= \dfrac{q^2 - 2rp}{r^2}$

$\therefore \Sigma\alpha^2\beta = \Sigma\alpha\beta \cdot \Sigma\alpha - 3\alpha\beta\gamma = -qp + 3r = 3r - qp$

11) If α, β, γ are the roots of the eq^n $x^3 + px^2 + qx + r = 0$ then find the value of $\Sigma\alpha^3$ and $\Sigma\alpha^2\beta^2$

$\Rightarrow$ $\Sigma\alpha = -p$, $\Sigma\alpha\beta = q$, $\alpha\beta\gamma = -r$

$\therefore \Sigma\alpha^3 = \Sigma\alpha \cdot \Sigma\alpha^2 - \Sigma\alpha^2\beta = \Sigma\alpha\{(\Sigma\alpha)^2 - 2\Sigma\alpha\beta\} - \{\Sigma\alpha\Sigma\alpha\beta - 3\alpha\beta\gamma\}$

$= -p(p^2 - 2q) - (-pq + 3r) = -p^3 + 2pq + pq - 3r$

$= 3pq - p^3 - 3r$

$\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma\Sigma\alpha$

$= q^2 - 2rp$

12) If α, β, γ are the roots of the eq^n $x^3+px^2+qx+r=0$ then find the value of $\Sigma\frac{1}{\alpha}$, $\Sigma\frac{1}{\alpha\beta}$, $\Sigma\frac{1}{\alpha^2}$

$\rightarrow$ $\Sigma\alpha=-p$, $\Sigma\alpha\beta=q$, $\alpha\beta\gamma=-r$; $\Sigma\frac{1}{\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}=-\frac{q}{r}$

$\therefore \Sigma\frac{1}{\alpha\beta}=\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}=\frac{\Sigma\alpha}{\alpha\beta\gamma}=\frac{p}{r}$

$\therefore \Sigma\frac{1}{\alpha^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\Sigma\alpha^2\beta^2}{(\alpha\beta\gamma)^2}=\frac{(\Sigma\alpha\beta)^2-2\alpha\beta\gamma\Sigma\alpha}{(\alpha\beta\gamma)^2}=\frac{q^2-2rp}{r^2}$

13) If α, β, γ are the roots of the eq^n $x^3-3px^2+3(p-1)x+1=0$ (i) then find the eq^n whose roots are $1-\alpha$, $1-\beta$, $1-\gamma$. Deduce that α, β, γ are all real.

$\Rightarrow$ Let, $y=1-\alpha$

$\therefore y=1-x$

or, $x=1-y$

$\therefore$ from (i), $(1-y)^3-3p(1-y)^2+3(p-1)(y-1)+1=0$

$\Rightarrow (1-y^3-3y+3y^2)-3p(1+y^2-2y)+(3p-3)(y-1)+1=0$

$\Rightarrow 1-y^3-3y+3y^2-3p-3py^2+6py+3py-3p-3y+3+1=0$

$\Rightarrow 1-y^3-3y+3y^2-3p-3py^2+y(-3+6p+3p-3)+4-3p-1=0$

$\Rightarrow -y^3+y^2(3-3p)+y(-6-9p)+3p-1=0$

$\Rightarrow y^3+3(p-1)y^2-y(-6-9p)+3p-1=0$ ——(ii)

$\Rightarrow y^3+3(p-1)y^2+3py(6+9p)+3p-1=0$

$\therefore$ eq^n (ii) is the required eq^n

Now putting $y=\frac{1}{x}$ i.e. $x=\frac{1}{y}$ into (i) we have,

$\frac{1}{y^3}-3p\frac{1}{y^2}+3(p-1)\frac{1}{y}+1=0$ ——(iii)

or, $y^3-3(1-p)y^2-3py+1=0$

the roots of eq^n (iii) are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Since, (ii) and (iii) are the same

$\therefore \frac{1}{\alpha}=1-\alpha$, $\frac{1}{\beta}=1-\beta$, $\frac{1}{\gamma}=1-\gamma$ ——(iv)

Let, if possible the roots of the eq^n are imaginary.

from the first eq^n of (iv) we have $\alpha^2-\alpha+1=0$ giving α imaginary

Similarly, from (iv) β and γ are imaginary.

Which is a contradiction.

$\therefore \alpha, \beta, \gamma$ are all real.